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The hypersonic laminar boundary layer near sharp compression and expansion corners

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The momentum integral method of Klineberg is shown to provide a good description of the major characteristics of two-dimensional laminar viscous–inviscid interactions at hypersonic speeds. Surface pressure and heat-transfer-rate measurements were made for sharp compression and expansion corners at Mach 12·2 and are compared with the theoretical predictions. The agreement is found to be good for attached, incipient and fully separated flows.

Some theoretical comparisons between methods based on the Klineberg formulation are made which suggest that the full boundary-layer equations are well described using integral methods that incorporate the energy equation. It is further shown that the properties associated with the stability of the governing differential equations are mathematical properties of the analytical model and should not be associated with any physical characteristics of the boundary layer.

A correlation of hypersonic, cold-wall, incipient separation data is also presented.

1. Introduction

In the design of any vehicle for high altitude hypersonic flight, the prediction of boundary-layer behaviour is of considerable importance. Of particular interest are the strong viscous–inviscid interactions which occur at the wing–flap junctions of control surfaces and on intake compression ramps.

At high Mach numbers, the development of the viscous layer can significantly modify the external flow field. The usual method of boundary-layer analysis, where viscous effects are considered as a perturbation on an already existing flow, is therefore inapplicable. However, the viscous interaction can be described by employing the boundary-layer equations together with a 'coupling' equation relating the development of the inner viscous flow to the outer inviscid flow. The governing partial differential equations can then be solved using finite-difference techniques or can be expressed in integral form and solved as ordinary differential equations.

One of the most successful integral methods is that of Lees & Reeves (1964) as extended by Holden (1965) and Klineberg (1968). The method employs the

Cohen–Roshotko (1956*a*) similarity solutions to provide the required relations between profile quantities and solves the integral equations of momentum, moment of momentum and energy, together with a coupling equation based on the integral form of the continuity equation. At low Mach numbers, good agreement with experiment has been obtained.

The main objective of this work is to evaluate the Klineberg method in predicting cold-wall interactions at Mach 12·2. Comparisons are made for two-dimensional laminar flow over sharp compression and expansion corners. Some analytic difficulties are also investigated.

2. Analysis

2.1. *Governing equations*

The governing equations are those of a steady, two-dimensional, compressible, laminar boundary layer. By applying the Stewartson (1949) transformation, the following integral form of these equations can be derived (Klineberg 1968):

$$H\frac{d\ln\delta_i^*}{dx}+\frac{dH}{dx}+(2H+1+S_wE)\frac{dl\ln M_e}{dx}=BC\frac{M_\infty}{M_e}\frac{P}{\delta_i^*Re}\quad\text{(momentum)},\qquad(1)$$

$$J\frac{d\ln\delta_i^*}{dx}+\frac{dJ}{dx}+(3J+2S_wT^*)\frac{d\ln M_e}{dx}=BC\frac{M_\infty}{M_e}\frac{R}{\delta_i^*Re}\quad\text{(moment of momentum)},\qquad(2)$$

$$T^*\frac{d\ln\delta_i^*}{dx}+\frac{dT^*}{dx}+T^*\frac{d\ln M_e}{dx}=BC\frac{M_\infty}{M_e}\frac{Q}{\delta_i^*Re}\quad\text{(energy)},\qquad(3)$$

where

$$B=(a_ep_e)/(a_\infty p_\infty),\quad C=(\mu/\mu_\infty)/(T/T_\infty),\quad S=(h_0/h_{0_e}-1),$$

$$Re=a_\infty M_\infty\delta_i^*/\nu_\infty,$$

$$\delta_i=\int_0^{\delta_i}dY,\quad \delta_i^*=\int_0^{\delta_i}\left(1-\frac{U}{U_e}\right)dY,$$

$$\theta_i=\int_0^{\delta_i}\frac{U}{U_e}\left(1-\frac{U}{U_e}\right)dY,\quad \theta_i^*=\int_0^{\delta_i}\frac{U}{U_e}\left(1-\frac{U^2}{U_e^2}\right)dY,$$

$$H=\frac{\theta_i}{\delta_i^*},\quad J=\frac{\theta_i^*}{\delta_i^*},\quad Z=\frac{1}{\delta_i^*}\int_0^{\delta_i}\frac{U}{U_e}dY,$$

$$R=2\delta_i^*\int_0^{\delta_i}\left[\frac{\partial}{\partial Y}\left(\frac{U}{U_e}\right)\right]^2dY,\quad P=\delta_i^*\left[\frac{\partial}{\partial Y}\left(\frac{U}{U_e}\right)\right]_{Y=0},$$

$$Q=-\delta_i^*\left[\frac{\partial}{\partial Y}\left(\frac{S}{S_w}\right)\right]_{Y=0},\quad E=\frac{1}{\delta_i^*}\int_0^{\delta_i}\frac{S}{S_w}dY,$$

$$T^*=\frac{1}{\delta_i^*}\int_0^{\delta_i}\frac{U}{U_e}\frac{S}{S_w}dY,$$

Y and U are the Stewartson transformed co-ordinate and velocity component respectively and the subscript ∞ denotes free-stream conditions, e the edge of the boundary layer and w the wall.

The development of the outer inviscid flow can be related to the boundary-layer growth by integrating the continuity equation across the boundary layer. This gives as the coupling equation

$$\frac{d\delta^*}{dx} - (\delta - \delta^*)\frac{d\ln(\rho_e u_e)}{dx} = \tan(\theta_e - \alpha_w),$$

where α_w is the local surface slope and θ_e is the streamline inclination at the edge of the boundary layer.

On transforming to Stewartson co-ordinates the coupling equation becomes

$$F\frac{d\ln\delta_i^*}{dx} + \frac{dH}{dx} + \frac{1+m_e}{m_e}S_w\frac{dE}{dx} + \frac{1}{1+m_e}\left[m_e F + 2H - \frac{1+2m_e}{m_e}Z\right]\frac{d\ln M_e}{dx} - \left[F + \frac{Z}{m_e}\right]\frac{d\ln p_e}{dx} = \frac{B(1+m_e)}{m_e(1+m_\infty)}\,\frac{\tan(\theta_e - \alpha_w)}{\delta_i^*}, \quad (4)$$

where $$F = H + \frac{1+m_e}{m_e}(1+S_w E), \quad m = \frac{\gamma-1}{2}M^2.$$

The streamline inclination θ_e at the edge of the boundary layer and the local static pressure p_e are related to the Mach number M_e by means of the model used to describe the outer inviscid flow. In the Klineberg formulation, the Prandtl–Meyer rule is used throughout the interaction except across corner shock waves where a significant entropy change occurs. This entropy change is considered to approximate that across the incident shock in an equivalent impinging shock system (Lees & Reeves 1964).

In the Klineberg formulation, the required relations between profile quantities (H, J, etc.) are provided by the Cohen–Reshotko (1956*a*) similarity solutions. Two parameters are used to describe these relations, one representing the velocity profile and the other the total enthalpy profile. The velocity profile parameter is taken to be the profile quantity H and the total enthalpy profile parameter, normalized with respect to S_w, is taken to be

$$b = -\frac{\partial}{\partial\eta}\left(\frac{S}{S_w}\right),$$

where η is the similarity variable of Cohen & Reshotko. Equations (1)–(4) thus reduce to a determinate set of equations with dependent variables M_e, δ_i^*, H and b.

It remains to specify the boundary conditions. However, unlike boundary-layer flows with a prescribed pressure gradient, viscous–inviscid interactions do not form well-posed initial-value problems. Through the mechanism of viscous interaction contained in the coupling equation, the solution is dependent on up-stream and downstream boundary conditions (Garvine 1968) and both must be specified. The upstream boundary conditions are provided by the strong or weak interaction asymptotic flat-plate flow solutions and the downstream boundary conditions by the Blasius flow solution.

2.2. *Stability of the governing differential equations*

In viscous–inviscid interaction problems the governing differential equations are usually unstable to downstream integration, i.e. if some parameter in the initial conditions is varied by an arbitrarily small amount each resulting solution ultimately diverges from the original one. Hence any given upstream boundary conditions will allow an infinite number of possible solutions, each of which satisfies a different downstream boundary condition. The instability of the governing differential equations thus provides the mechanism for the large-scale upstream influence that is observed in many hypersonic boundary-layer flows.

However, not all of solution space exhibits this unstable behaviour and in certain regions the governing differential equations are stable. In these regions the solution is insensitive to downstream boundary conditions and there can be no upstream influence. In some hypersonic boundary-layer flows (e.g. expansions) this is not a physically unrealistic situation. However, if the effects of upstream influence are to be preserved (e.g. for the compression corner), it is necessary to allow a 'jump' to the unstable state at the beginning of interaction (see Klineberg 1968).

Sometimes the solution passes smoothly from the unstable state to the stable state or vice versa. The transition between states occurs at the Crocco–Lees critical point (Crocco & Lees 1952), a singularity associated with the vanishing of the determinant of the governing set of equations. The locus of critical points (the 'critical boundary') thus separates regions exhibiting unstable behaviour from those exhibiting stable behaviour.

2.3. *Solution procedure*

The upstream boundary conditions are determined and, depending on the stability of the governing differential equations and on the expected degree of upstream influence, a 'jump' is or is not applied. Using a fourth-order Runge–Kutta method, the governing differential equations are then integrated in the downstream direction. If the resulting solution trajectory does not satisfy the downstream boundary conditions, a parameter in the upstream boundary conditions is perturbed and another solution trajectory determined. This parameter is then iterated upon until the downstream boundary conditions are satisfied.

If the Crocco–Lees critical point is encountered during integration, an extrapolation procedure is adopted to allow the solution to pass smoothly through this singularity.

3. Experimental study

The experimental study was conducted in the Imperial College no. 1 gun tunnel using a Mach 12·2 contoured nozzle. The details of this facility have been described by Stollery, Maull & Belcher (1960). Stagnation conditions $p_0 = 1600$ psia and $T_0 = 1300\,^{\circ}$K were used, which gave a free-stream unit Reynolds number of $0{\cdot}87 \times 10^5\,\text{in.}^{-1}$. The useful running time was approximately 20 ms.

Two models were used in the experiments. In the compression-corner case, the model consisted of a 6 in. chord, sharp, flat plate at zero incidence, hinged at its downstream end to a 5 in. chord flap. Tests were made at flap angles of 8°, 10° and 12°. The expansion-corner configuration was chosen to avoid any trailing-edge shock, which might have induced separation. The model consisted of a 5 in. chord, flat plate at zero incidence with an 8 in. chord, leading-edge flap. Flap angles of $-5°$, $-10°$ and $-20°$ were used, providing a wide range of flow conditions in the approach to the expansion corners. The span of the models was 5 in. and the leading-edge thicknesses were about 0·002 in.

The models were instrumented with thin-film, platinum, heat-transfer gauges and static-pressure orifices both along and off the centre-line. A description of the heat-transfer-rate measuring equipment, which employs an analog network to convert the surface temperature signal into the heat-transfer rate, has been given by Hunter (1969). Surface pressures were measured using both C.E.C. (4/326) 0–10 psia unbonded strain gauges and Pitran (PTM3) 0·25 psid pressure-sensitive transistors, connected by tubing to orifices $\frac{1}{16}$ in. in diameter. The C.E.C. (4/326) gauge, which has a sensitivity of 4 mV/psi, is limited by circuit noise at pressure levels below about 0·05 psia. Measurement of the low pressures was made using the Pitran (PTM3) transducers, which have a nominal sensitivity of 4V/psid.

Focused schlieren photographs were taken at any instant during the run using a conventional single-spark single-pass optical system. Typical photographs of the flows over the compression and expansion corners are shown in figures 1 (*a*) and (*b*) (plate 1). The edge of the white line next to the surface on the schlieren photographs is located near the edge of the boundary layer and gives a measure of the boundary-layer thickness.

It is, of course, essential that the centre-line flows be two-dimensional and laminar if a direct comparison is to be made with theory. Laminar flow was confirmed by the appearance of a well-defined white line in the boundary layer together with steady heat-transfer traces. Previous experiments made in the same facility have shown that transition to turbulent flow is associated with noisy heat-transfer traces. The schlieren photographs, however, indicate weak disturbances over the compression flaps. These disturbances may be due to the effect of viewing through the thick turbulent mixing layer at the edge of the jet. Two further tests were made to confirm laminar flow. Schlieren photographs were taken of the flow over the 12° compression-corner model, at a fixed value of wall-to-total temperature ratio and at two values of the Reynolds number. It was seen that the separation point moves upstream as the Reynolds number increases, which is in agreement with the trend for a laminar boundary layer found by other workers. A Pitot profile was taken downstream of the peak pressure position on the 10° flap and the profile was compared with the theoretical zero-pressure-gradient result of van Driest (1952). The comparison indicated that the boundary layer was laminar.

No direct tests for two-dimensional flow were made in the present study, while there is insufficient information in the literature to infer two-dimensionality from other work. However, the necessary condition of uniform spanwise pressure and heat-transfer distributions was found to be satisfied.

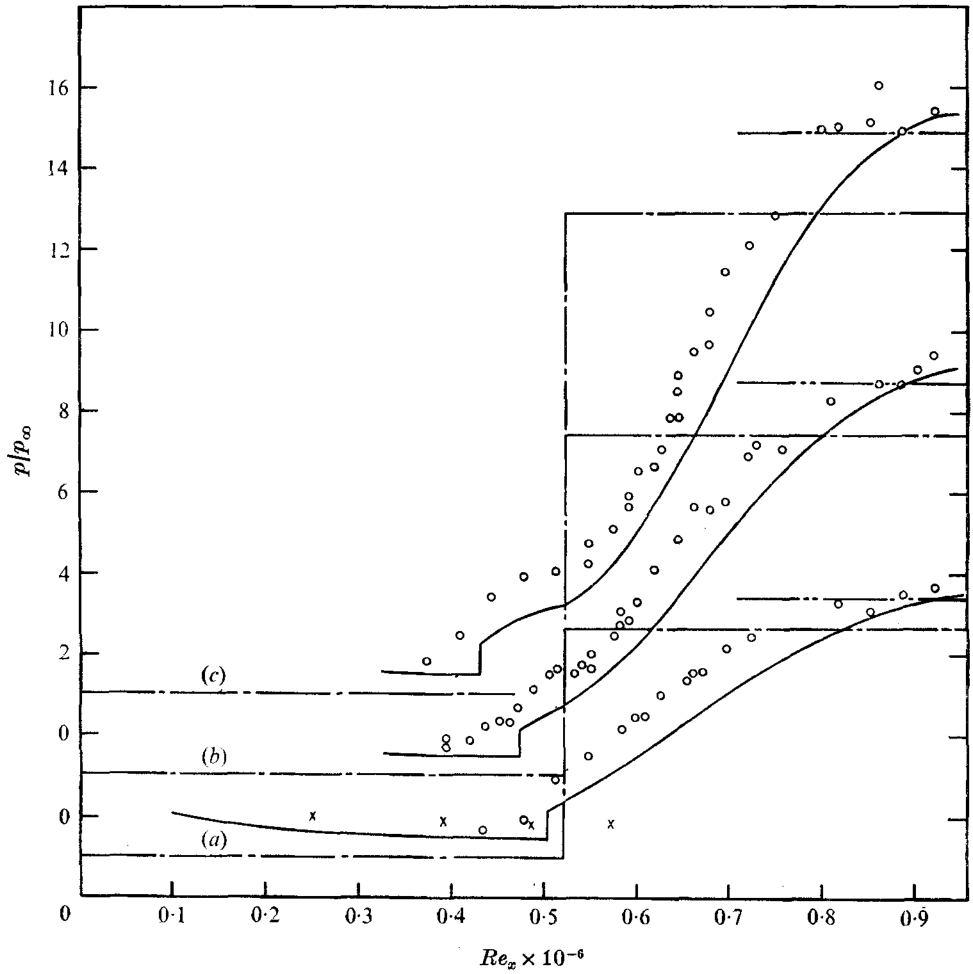

FIGURE 2. Pressure distribution on the compression-corner model; $M_\infty = 12{\cdot}2$, $Re_\infty = 0{\cdot}87 \times 10^5$ in.$^{-1}$, $T_w/T_0 = 0{\cdot}22$. ×, data from a flat plate at zero incidence; o, compression-corner data; —, Klineberg method; — - —, inviscid; —· · —, pressure across the two-shock system. (*a*) $\alpha = 8°$. (*b*) $\alpha = 10°$. (*c*) $\alpha = 12°$.

For the effect of finite ramp length on the compression-corner flow interactions, the present data from the 10° expansion-corner model indicate that the rapid expansion associated with the trailing edge propagates about five boundary-layer thicknesses upstream.

4. Comparison between theory and experiment

For the cases considered enough data are available to provide a unique solution to the analytical problem. Hence the method of Klineberg was used in a purely predictive capacity and no matching of theoretical and experimental results was made.

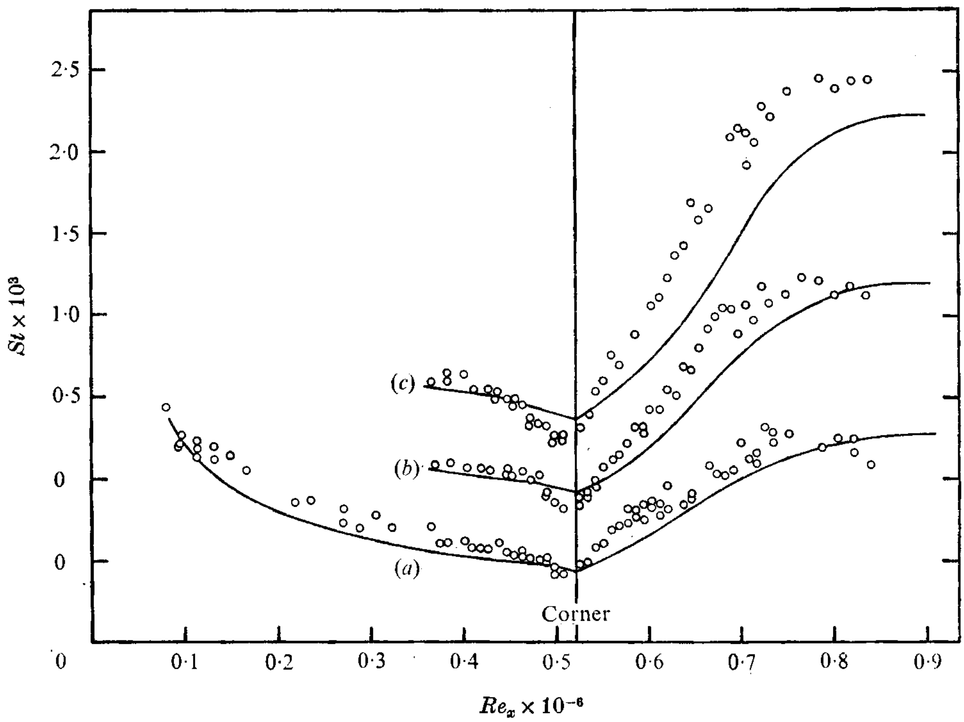

FIGURE 3. Heat-transfer-rate distribution on the compression-corner model; $M_\infty = 12{\cdot}2$, $Re_\infty = 0{\cdot}87 \times 10^5\,\text{in.}^{-1}$, $T_w/T_0 = 0{\cdot}22$. o, present experimental data; ——, Klineberg method. (*a*) $\alpha = 8°$. (*b*) $\alpha = 10°$. (*c*) $\alpha = 12°$.

4.1. *Compression corner*

The predicted distributions of surface pressure and heat-transfer rate on the 8°, 10° and 12° compression-corner models are compared with the experimental data in figures 2 and 3, where the three curves and sets of points are staggered vertically.

The general agreement between theory and experiment is good. In the flat-plate weak viscous interaction region, the experimental data lie slightly above the theoretical results. This is probably due to the effect of the small leading-edge bluntness, which is not accounted for by the theory. Over the flap, the boundary-layer characteristics respond more rapidly to the disturbance generated at the corner than is predicted by the theory. This probably results from the inexact nature of the Lees–Reeves shock interaction model. It cannot be assumed that the lag results from the propagation of flow disturbances through the boundary layer along Mach lines (no such lag is observed, for example, in the expansion-corner comparisons). The maximum values of the pressure ratio and heat-transfer rate are accurately predicted. The final pressure ratio lies above the inviscid value because the stagnation pressure losses through the shock system are not as great as those given by inviscid theory (see Georgeff 1972). As shown in figure 2, the final pressure ratio can be better estimated by considering the two-shock system formed by the leading-edge and corner shock waves (see Bloy 1973).

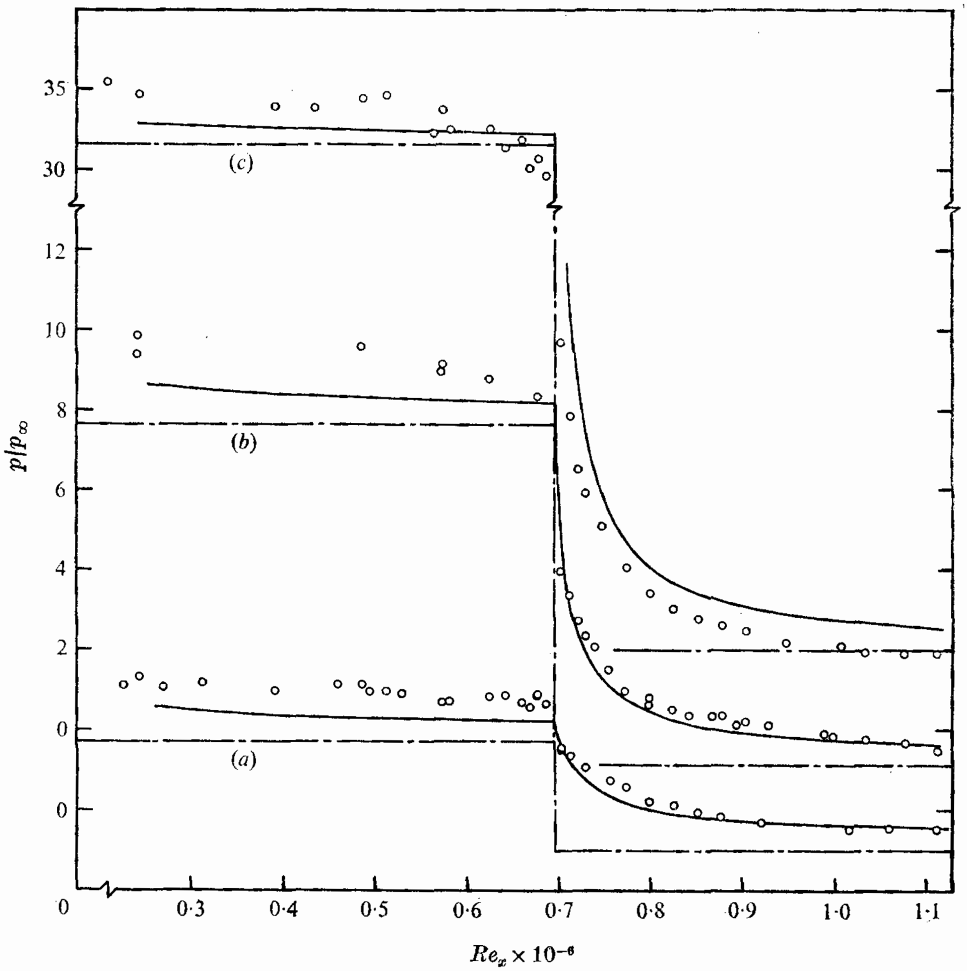

FIGURE 4. Pressure distribution on the expansion-corner model; $M_\infty = 12{\cdot}2$, $Re_\infty = 0{\cdot}87 \times 10^5$ in.$^{-1}$, $T_w/T_0 = 0{\cdot}22$. ○, present experimental data; ——, Klineberg method; — — —, inviscid. (a) $\alpha = -5°$. (b) $\alpha = -10°$. (c) $\alpha = -20°$.

However, the respective shock wave angles must be known (in this case, using the measured flow deflexion at the edge of the boundary layer) and thus the technique is limited in its predictive capacity. Note that the good description of stagnation pressure losses and final pressure ratio provided by the Lees–Reeves model of shock wave interaction derives from the approximations made and not from accurate modelling of the overall shock system.

4.2. *Expansion corner*

The predicted distributions of surface pressure and heat-transfer rate on the 5°, 10° and 20° expansion-corner models are compared with the experimental data in figures 4 and 5, where the three curves and sets of points are staggered vertically.

Again the general agreement with experiment is good. Upstream of the corner, the heat-transfer rate is well described except at the largest inclination

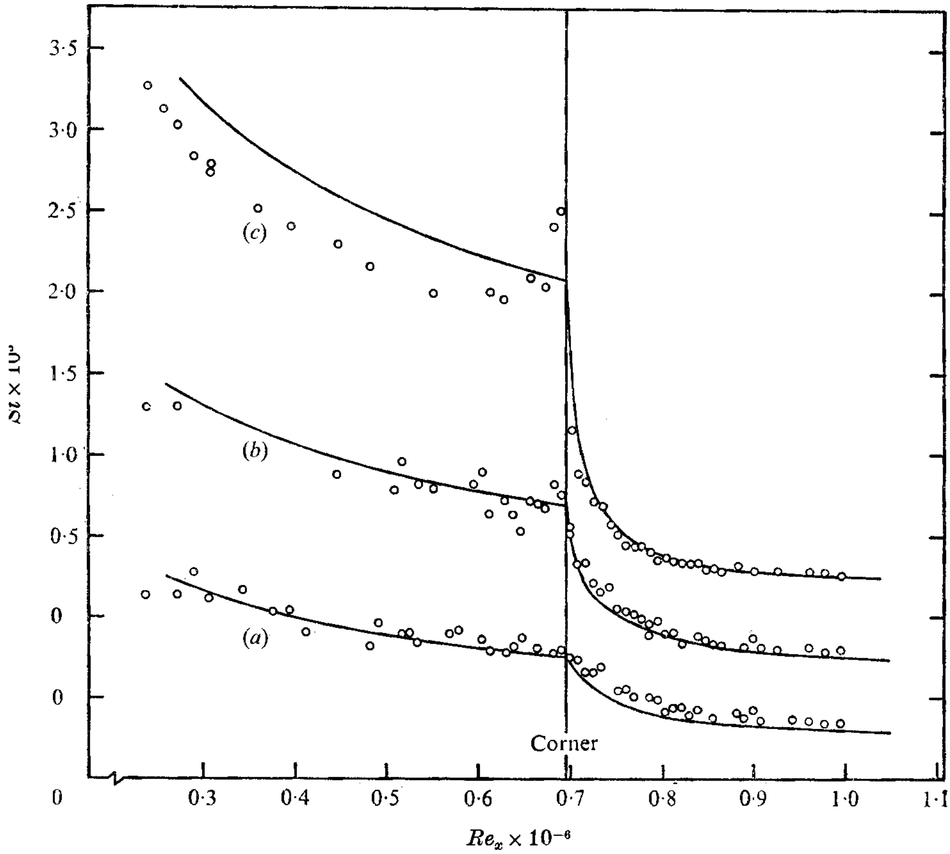

FIGURE 5. Heat-transfer-rate distribution on the expansion-corner model; $M_\infty = 12{\cdot}2$, $Re_\infty = 0{\cdot}87 \times 10^5$ in.$^{-1}$, $T_w/T_0 = 0{\cdot}22$. o, present experimental data; ——, Klineberg method. (*a*) $\alpha = -5°$. (*b*) $\alpha = -10°$. (*c*) $\alpha = -20°$.

angle, where it is overpredicted by approximately 10–15 %. The small underprediction of the pressure ratio could again result from the effect of the small leading-edge bluntness. Downstream of the corner, the agreement with experiment is very good. No upstream influence is predicted as the governing differential equations are stable. This agrees well with experiment, except at the largest inclination angle, where the extent of upstream influence is of the order of 10 boundary-layer thicknesses.

In the Klineberg method, normal pressure gradients are neglected. The good agreement between theory and experiment, despite the existence of large normal pressure gradients in the real flow, therefore suggests that the *effect* of normal pressure gradients within the hypersonic boundary layer is small.

4.3. *Correlation of hypersonic, cold-wall, incipient separation data*

One of the objectives in the study of shock–boundary-layer interactions is the prediction of incipient separation. In two-dimensional flow, this is defined as the condition such that there exists one point only in the interaction region with zero skin friction. Since skin friction has proved difficult to measure,

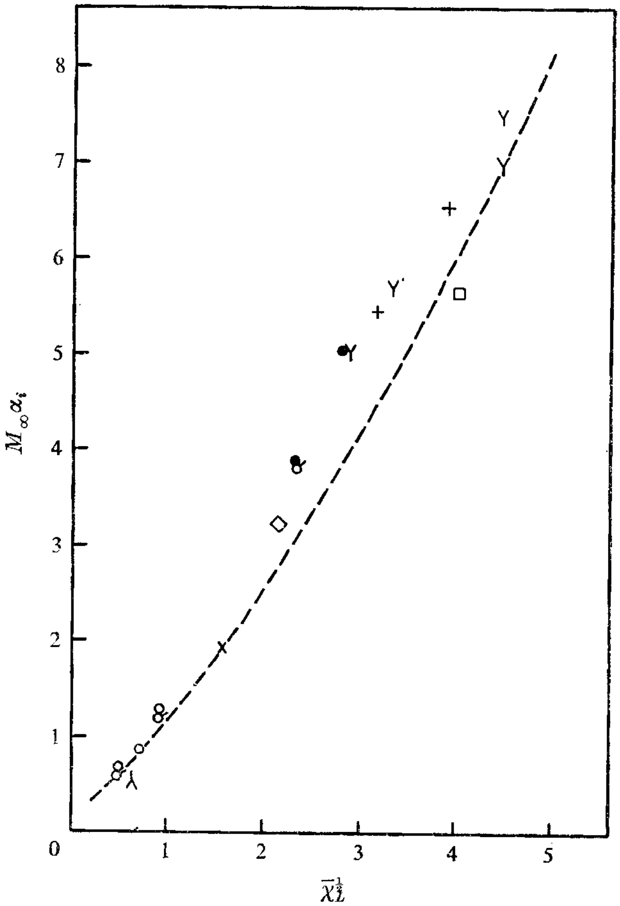

FIGURE 6. Correlation of hypersonic, cold-wall, incipient separation data. Experimental data: ⅄, Ball & Korkegi (1968); o, Needham (1965); ◇, Nielsen (see Richards & Enkenhus 1970); ×, present study; ●, Stollery (unpublished); Y, Holden (1967, 1971); □, Harvey (1968); +, Miller *et al.* (1964). — —, Klineberg method ($T_w/T_0 = 0{\cdot}2$); symbols with flags indicates incident shock data.

experimenters have resorted to the detection of incipient separation from surface-pressure and heat-transfer-rate measurements and from schlieren photographs. The first appearance of an inflexion point in the pressure distribution, a change in the heat-transfer distribution from a cusp-like minimum to a rounded concave minimum, or the appearance of a separation shock have been taken to indicate the incipient separation condition. These conclusions have been supported by the incipient separation data of Holden (1971).

Using the Klineberg formulation, the parameters governing viscous–inviscid interactions can be shown (Georgeff 1972) to be $M_{\infty}\alpha_w$, $\bar{\chi}_L$ and S_w, where L is a characteristic length (in this case, the distance from the leading edge to the hinge line). It thus follows that the incipient separation condition (denoted by the subscript i) for geometrically similar bodies must be of the form

$$(M_{\infty}\alpha_w)_i = f(\bar{\chi}_L, S_w). \quad (5)$$

All the available hypersonic, cold-wall, incipient separation data are plotted in the above form in figure 6, where they are compared with the theoretical results obtained at $S_w = -0{\cdot}8$. As only cold-wall cases were considered, the dependence on S_w has been neglected. The incipient separation condition is clearly well described by the parameters given above and the agreement between theory and experiment is good.

5. Analytical considerations

Some of the problems encountered in the analysis of viscous–inviscid interactions can be resolved by comparing methods based on the Klineberg approach.

5.1. *Description of the methods*

The methods are briefly described below. All use the same model of the flow and essentially the same approach, differing only in the set of governing equations employed, the assumed relations between profile quantities, and the number of parameters left free.

(i) The method of Lees & Reeves (1964) is similar to the method of Klineberg except that the enthalpy profile is considered to be a function of the velocity profile. The energy integral equation is therefore not required. Using the Cohen–Reshotko similarity solutions to provide the required universal relations, all the profile quantities are expressed as functions of the profile parameter H. The resulting equations provide a determinate set of equations for the three unknowns M_e, δ_i^* and H.

(ii) Stollery & Hankey (1970) modified the method of Klineberg by replacing the exact relation between the profile quantities J and H as determined from the Cohen–Reshotko similarity solutions by a linear approximation, viz.

$$J = KH, \tag{6}$$

where K is a constant. The governing equations remain unchanged, although they can be simplified.

(iii) Another modified form of the Klineberg method is obtained by replacing the moment of momentum integral equation (2) by the momentum equation at the wall, which in Stewartson co-ordinates is (Georgeff 1972)

$$\frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{I}{\delta_i^* Re}, \tag{7}$$

where

$$I = -\frac{\delta_i^{*2}}{(1+S_w)} \frac{\partial^2}{\partial Y^2} \left[\frac{U}{U_e}\right]_{Y=0}.$$

As before, the required universal relations are provided by the Cohen–Reshotko similarity solutions, and all the profile quantities (including I) are expressed as functions of the profile parameters H and b. The resulting equations again provide a determinate set of equations in the four unknowns M_e, δ_i^*, H and b. This method is closely related to the methods of Cohen & Reshotko (1956*b*), Bray, Gadd & Woodger (1960) and Curle (1961), although many of the simplifying assumptions employed in these earlier methods are dropped.

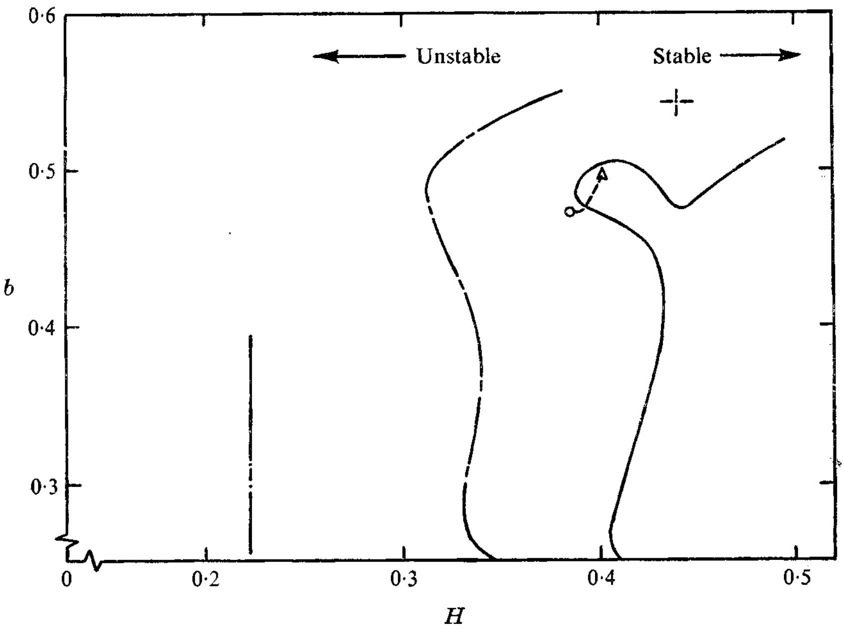

FIGURE 7. Critical boundary for hypersonic flow at $S_w = -0{\cdot}8$. -+-, Lees & Reeves (1964); — — —, Klineberg (1968); ——, Stollery & Hankey (1970) or Georgeff (1972); - - -, flat-plate solution; △, strong interaction; o, weak interaction (Blasius solution); — · — · —, separation-point solution.

At hypersonic speeds the critical boundary associated with each of the methods described is independent of Mach number and is plotted in H, b space in figure 7 for $S_w = -0{\cdot}8$. It is observed that the position of the critical boundary is strongly dependent on the formulation of the problem.

The sensitivity of the position of the critical boundary to changes in the profile quantities is also shown by comparison of the essentially identical methods of Klineberg (1968) and Holden (1965). Holden's use of slightly different profile quantity relations (due to a different order of curve fit) shifts the critical boundary to the right, so that it lies between the weak interaction solution and the strong interaction solution. Hence, if Holden's formulation is used the flat-plate weak interaction solution is unstable whereas if Klineberg's is used it is stable.

In terms of computational efficiency, the methods described are virtually indistinguishable.

5.2. *Comparison with experiment*

In figures 8 (*a*) and (*b*), the methods described are compared with the experimental data for the 12° compression corner, It is clear that the Lees–Reeves method, where the energy equation is not employed, is inadequate for describing the distribution of the heat-transfer coefficient (and similarly the skin-friction coefficient). However, the other modifications make little difference to the solutions obtained and all provide good agreement with experiment. In this case, the modifications (ii) and (iii) do not require a jump to initiate interaction nor do they encounter the Crocco–Lees critical point.

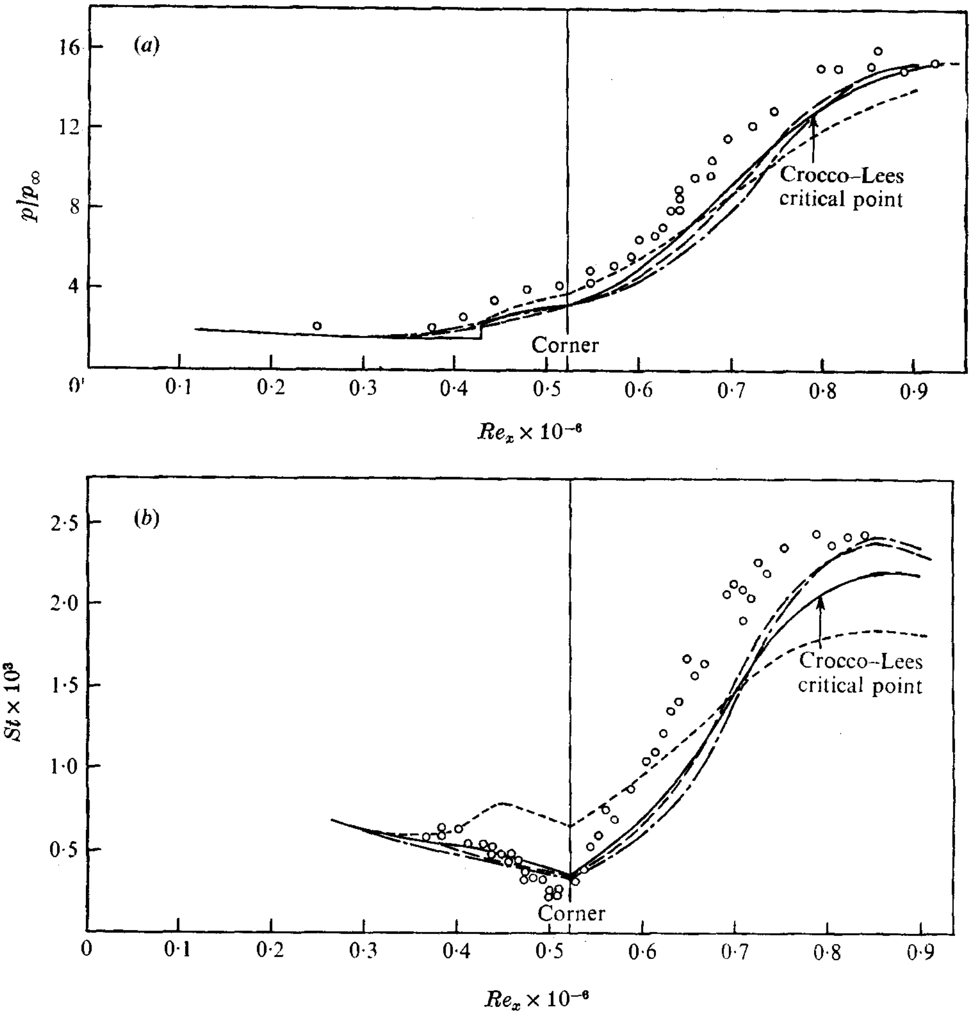

FIGURE 8. (*a*) Pressure and (*b*) heat-transfer-rate distribution on compression corner at $\alpha = 12°$; $M_\infty = 12{\cdot}2$, $Re_\infty = 0{\cdot}87 \times 10^5$ in.$^{-1}$, $T_w/T_0 = 0{\cdot}22$. – – – –, Lees & Reeves (1964); ——, Klineberg (1968); — —, Stollery & Hankey (1970); — – —, Georgeff (1972); o, present experimental data.

5.3. *Physical significance of changes in stability*

Weinbaum & Garvine (1969) demonstrated that the stability of the governing differential equations was related to the Mach number profile within the viscous layer. The 'throat' station, or critical point, was found to occur where the viscous layer changed, in some mean sense, from subsonic to supersonic with a corresponding change in the stability of the governing differential equations. However, the analysis was limited to free shear layers, and normal pressure gradients, which could well influence the behaviour of the solution, were neglected.

The most rigorous analysis of boundary-layer flows that satisfy a zero-slip boundary condition has been given by Stewartson & Williams (1969). The boundary layer was divided into a viscous inner layer and an outer inviscid layer in which the normal pressure gradients were included. It was found that an eigen-solution departing towards separation existed under all flow conditions, and no behaviour analogous to a change in stability was observed. Hence, although it may be possible to attribute some physical significance to changes in stability in free shear layers (see also Baum & Denison 1967), this does not appear to be the case in boundary-layer flows.

The results of the preceding comparisons enforce this view. It was shown that the stability is strongly dependent on the choice of the governing differential equations and on the assumed relations between profile quantities, whereas the solutions obtained were found to be relatively insensitive to the formulation. Furthermore, it was shown that the imposition of a jump does not significantly affect the solution except in the vicinity of the jump. These results imply that the distinction between stable and unstable states is not a physical characteristic of the boundary layer but, on the contrary, is a result of the formulation of the problem.

It thus appears that, in boundary-layer flows, efforts to ascribe physical significance to changes in stability are ill founded (see also Stollery & Hankey 1970; Georgeff 1971). Hence the analogy between stable–unstable (or 'super-critical–subcritical') states of the boundary layer and supersonic–subsonic flows (Crocco 1955) must be considered weak, at least from a physical standpoint. Procedures adopted for passing from the stable to the unstable state [e.g. a 'jump' to separation (Crocco 1955; Holden 1965), a 'jump' based on the conservation laws (Klineberg 1968) or subjection to a strong negative pressure gradient (Reyhner & Flügge-Lotz 1966)] are, in the light of the above, purely expedients and also lack physical significance. Furthermore, the constraints imposed at the Crocco–Lees critical point and the velocity profile critical point (see Georgeff 1971) are mathematical in nature and should not be associated with physical properties of the boundary layer (cf. Murphy 1969; Shamroth 1969). Even in viscous wake flows, where the analysis of Weinbaum & Garvine is perhaps valid, care must be taken in ascribing physical significance to the stability of the governing equations of any approximate method.

5.4. *Methods of solution*

The choice between approaches exhibiting differences in type (stability) depends on the problem and its application. Computational efficiency is enhanced by using stable governing differential equations, while accurate modelling of upstream influence requires unstable governing differential equations.

For methods based on the Klineberg formulation, the preceding comparisons indicate that the solutions obtained are relatively insensitive to the initial stability of the governing differential equations. For these methods differences in stability have only a local effect, and all provide a good description of the overall properties of viscous interaction phenomena.

5.5. '*Unhooking*' *of the pressure gradient*

If, using the Cohen–Reshotko similarity solutions, the local pressure gradient is fully determined by ('hooked to') the local values of the dependent variables, M_e, δ_i^*, H and b, then it cannot reach zero in the pressure plateau region. This was cited by Lees & Reeves (1964) as the major reason for replacing the momentum equation at the wall with the moment of momentum equation. However, while 'hooking' prevents the pressure gradient from reaching zero in the pressure plateau region, it does not prevent the pressure gradient from becoming exceedingly small. Thus it cannot be assumed *a priori* that 'hooking' of the pressure gradient will result in poor agreement with experiment. This is borne out in the preceding comparisons. The linearization of the relationship between J and H (modification (ii)) and the use of the momentum equation at the wall (modification (iii)) 'hooks' the pressure gradient to the local values of the dependent variables. Yet both methods provide as good, if not better, agreement with experiment in the pressure plateau region as do the 'unhooked' methods of Lees & Reeves and Klineberg.

5.6. *Validity of the integral approach*

The preceding comparisons show that it is necessary to employ the energy integral equation in order to describe accurately non-adiabatic viscous–inviscid interaction phenomena. Otherwise the choice of governing differential equations or a set of relations between profile quantities is not critical. This suggests that the partial differential equations are well described using integral methods that incorporate the energy equation. This conclusion is supported by the results of Murphy (1969), where the solutions obtained from methods based on the full partial differential equations and on integral methods are shown to be in good agreement with one another. The results also put the Klineberg method on firmer ground, as conclusive support for the choice of integral equations and profile functions is not necessary.

It is to be noted that many of the early objections raised by Brown & Stewartson (1969) and Stewartson & Williams (1969) to the Klineberg method do not arise in the present formulation. The difficulty in passing through separation using the profile parameter a is removed by using the profile parameter H. Computation, far from being complex, is relatively simple and is not substantially more difficult for non-adiabatic flows. Furthermore, any doubts about the dropping of the momentum equation at the wall are allayed by the excellent agreement between the solutions obtained using the method of Klineberg and those obtained using the modification (iii). The method is 'irrational' in Van Dyke's (1964, p. 2) sense in that it is not exact in any known limit. However, this does not deny the utility of such an approach.

6. Conclusions

The integral method of Klineberg has been compared with data obtained for cold-wall compression and expansion corners at Mach 12·2. The agreement between theory and experiment was found to be good for attached, incipient and fully separated flows. These results, together with the results of earlier comparisons at low Mach number, indicate that the method of Klineberg provides a good description of the major features of viscous–inviscid interaction phenomena at both supersonic and hypersonic speeds under adiabatic and non-adiabatic wall conditions.

An extensive theoretical evaluation of the method has been made, thus establishing the validity of the integral approach. It has been shown that it is necessary to 'uncouple' the velocity and enthalpy profile parameters and thus to employ the energy integral equation in order to describe accurately non-adiabatic viscous interactions. Otherwise the choice of governing differential equations or profile quantities is not critical. Contrary to the views of earlier investigators, it has been shown that it is not necessary to 'unhook' the local pressure gradient from the local values of the dependent variables in order to obtain good agreement with experiment. It has also been shown that the properties associated with the stability of the governing differential equations (i.e. 'subcritical' and 'supercritical' behaviour, the necessity to initiate compressive interaction by means of a 'jump', and the appearance of the Crocco–Lees critical point during integration) are mathematical properties of the analytical model and should not be associated with any physical characteristics of the boundary layer.

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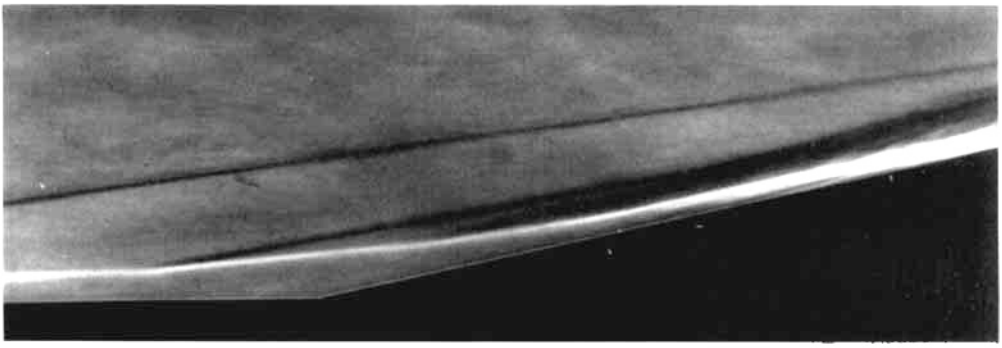
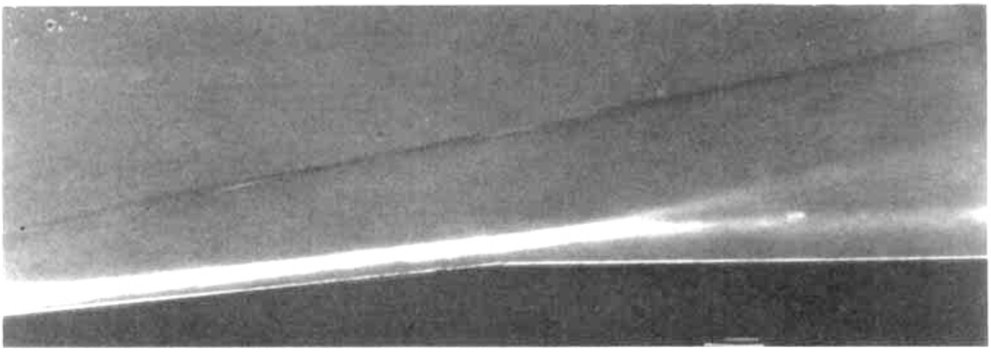


(*a*)



(*b*)

FIGURE 1. Schlieren photograph of the flow over (*a*) the 12° compression corner and (*b*) the 5° expansion corner; $M_\infty = 12{\cdot}2$, $Re_\infty = 0{\cdot}87 \times 10^5$ in.$^{-1}$, $T_w/T_0 = 0{\cdot}22$.